Physica Scripta. Vol. 46, 63–64, 1992.

A Heuristic Prediction of the Number of Solitons that are Excited by an Arbitrary Potential

Karl E. Lönngren[a, b] and Paul Hansen[b]

The University of Iowa, Iowa City, Iowa 52242

and

Tom Kuusela

Department of Applied Physics, University of Turku, 20500 Turku, Finland

Received November 4, 1991; accepted February 4, 1992

Abstract

The number of Korteweg–deVries solitons that are excited from an arbitrary potential are determined using a simple intuitive model. The results are compared with a numerical study of a nonlinear transmission line. The technique can be extended to nonlinear Schrödinger solitons.

The number of Korteweg–deVries (KdV) solitons that are excited due to an arbitrary excitation potential can be computed using inverse scattering theory. This technique relates this number to the number of bound states that are found in solving the linear Schrödinger equation where the excitation potential is the potential well. The technique is well described in several texts and will not be reproduced here [1]. It has been verified experimentally for ion acoustic solitons in a plasma by Hershkowitz *et al.* [2] and Ikezi [3]. The mathematical procedure is, however, rather complicated and at first brush, may not be very intuitive.

The purpose of the present note is to suggest a graphical model for the soliton excitation that predicts the number of excited solitons as a function of the size of the excitation potential. The model is based on two fundamental KdV soliton properties: (a) The product of the amplitude of the soliton A and the square of its width W is equal to a constant

$$A \cdot W^2 = \text{constant}. \tag{1}$$

The value of the constant depends upon the particular values of the coefficients that appear in the KdV equation. (b) Solitons do not overlap since their velocities are amplitude dependent.

From the first property, we write that

$$\sqrt{A} \cdot W = K \tag{2}$$

where K is a constant. Rather than employ the exact sech $^2\zeta$ profiles for the KdV solitons, let us consider them to be pulses. The product given in (2) implies an area of a rectangular pulse. The excitation potental whose amplitude is B and width is L will have the same dimensions as (2) if we write it as

$$\sqrt{B} \cdot L \tag{3}$$

The second property permits us to ascertain the number of solitons that will be excited. In order to effect this computation, we draw the product defined in (3) as a large rectangle in Fig. 1(a). In addition, we draw the product defined in (2) as a series of small rectangles. The number of solitons that will be excited is equal to the number N of small rectangles that will fit into the large rectangle plus one.

$$\#\ \text{of excited solitons} = N + 1 \tag{4}$$

If there is excess space within the large rectangle but not enough to accommodate an additional small one, an additional soliton will not be excited. Radiation will appear following this surplus. The number of excited solitons will increase in discrete steps that are proportional to the width and the square root of the amplitude of the excitation potential.

In order to test this hypothesis, we numerically investigated a Toda lattice [4]. The electrical analogy of the Toda lattice consists of distributed series inductors and shunt nonlinear capacitors. In the limit of continuous elements, this transmission line can be described by a KdV equation [5]. The rectangular input pulse of the transmission line can be regarded as an excitation potential. The results of this simulation are shown in Fig. 2. The simulation shows the

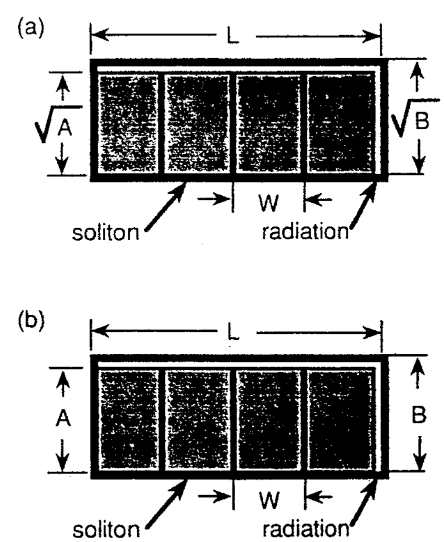

Fig. 1. Soliton excitation model. Each shaded rectangle represents an excited soliton. The excess unshaded region represents radiation. (a) KdV solitons. (b) NLS solitons

[a] Department of Electrical and Computer Engineering.
[b] Department of Physics and Astronomy.

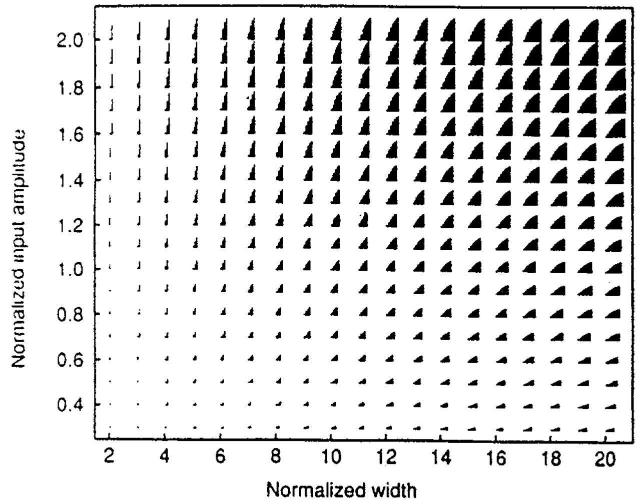

Fig. 2. The number of the solitons and the normalized amplitude of each one as a function of the normalized width and the normalized amplitude of the rectangular input pulse. Each line in a cluster is a soliton and the height of each line represents the amplitude

soliton excitation as either the amplitude or the width of the excitation signal are changed. At a fixed width, the number of solitons scales in proportion to the square root of the amplitude. At a fixed amplitude, the number of solitons scales in proportion to the width. Both results, verified also experimentally, are in agreement with the model. Note that an additional soliton is launched at discrete integer values in the parameter space.

A heuristic model has been found to determine the number of KdV solitons that will be excited due to an arbitrary excitation potential. Also, an inverse scattering analysis of box initial conditions for the KdV equation results in the same quantization rule determining soliton production, i.e. the rule given by (4). A similar heuristic and inverse scattering analysis for the nonlinear Schrödinger (NLS) equation requires that the boxes given in (2) and (3) have the square root signs be removed. See Fig. 1(b). We speculate that there may exist a fundamental quantization rule associated with these observations that awaits rigorous proof.

We will report the details in a future paper, but summarize the inverse scattering results as follows. For the KdV equation, the associated scattering problem is the Schrödinger equation of quantum mechanics [6] for which we consider a box potential as a model of the initial pulse. Solitons correspond to the bound states of this associated scattering problem. The bound state spectrum for this problem has already been obtained graphically [7], with the result that $A \cdot W$ does indeed determine the number of the bound states (solitons) according to rule (4). We have performed a similar calculation for the threshold for soliton production for the NLS equation, using the appropriate associated scattering problem, and have found $A \cdot W$ determines soliton production in this instance. These results are interesting because these combinations of A and W are constant conformal invariants of the respective nonlinear equations [8].

This work was sponsored in part by the N.S.F. Grant No. ECS 90-06921 and the NASA Grant No. NAGW-970.

References

1. e.g. Lamb, G. L. Jr., "Elements of Soliton Theory" (Wiley–Interscience, New York 1980), pp. 113–132.
2. Hershkowitz, N., Romesser, T. and Montgomery, D., Phys. Rev. Lett. **29**, 1586 (1972).
3. Ikezi, H., Phys. Fluids **16**, 1668 (1973).
4. Kuusela, T. and Hietarinta, J., Rev. Sci. Instrum. **62**, 2266 (1991).
5. e.g. Lonngren, K. E., in: "Solitons in Action" (Edited by K. E. Lonngren and A. C. Scott) (Academic, New York 1978), pp. 127–152.
6. Ablowitz, M. J., Kaup, D. J., Newell, A. C. and Segur, H., Stud. Appl. Math. **LIII**, 249 (1974).
7. Schiff, L., "Quantum Mechanics" (McGraw-Hill, New York 1968), 3rd edn, p. 39.
8. e.g. Lonngren, K. E., Proc. Ind. Acad. Sci. **86A**, 125 (1977).